Satellite Motion about an Oblate Earth

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A general theory of the method of averaging is used to study the effect of the Earth's oblateness on the motion of an artificial satellite. The first-order harmonic J_2 and the second-order harmonics J_3 and J_4 are included in the analysis. This paper illustrates a thorough and straightforward application of a second-order method of averaging to the orbital motion of a satellite about an oblate Earth with the well-known conventional orbital elements as the time-dependent variables. The first-order periodic variations together with the second-order averaged and secular variations are derived for all the six orbital elements. Essential transformations and initialization procedures are also outlined.

I. Introduction

IT is well known that for a close-Earth satellite, the inclusion of the perturbations due to the Earth's oblateness and the atmospheric drag is of vital importance to predict the lifetime and the trajectory of the satellite accurately. Except for the pioneer work of Brouwer,[1] Lane,[2] and Cranford,[3] relatively little analytical work has been done in studying the joint effects of the Earth's oblateness and atmospheric drag on an orbiting satellite. Although this study will not develop a joint treatment, it does present a second-order theory of an artificial satellite about an oblate Earth. This theory establishes a foundation for the joint treatment of the Earth's oblateness, atmospheric drag, and other perturbations (e.g., extra-terrestrial gravitation) through application of the general theory of the method of averaging. This approach is feasible not only because the transformations involved need not be canonical, but also because it provides a rigorous, systematic and straightforward procedure for studying a dynamical system perturbed by conservative and/or nonconservative forces. Further, this approach is also general in the sense that both the Von Zeipel method and the two-variable asymptotic expansion procedures are particular cases of the method of averaging.[4,5] Specifically, the present task is to develop a second-order theory which includes the effects contributed by the second, third, and fourth harmonics of the Earth's gravitational potential by applying the general theory of the method of averaging with the conventional orbital elements as its variables.

General perturbation theories applied to a close-Earth satellite under the influence of the oblateness of the Earth have been published by a number of authors to predict satellite motions. To name a few, second-order treatments have been given by Kozai,[6] Brouwer,[7] Blitzer,[8] Kyner,[9] Lorell et al.,[10] and Arsenault et al.[11] These and a large number of other theories and their results have been compiled and compared in Ref. 11. Kozai[6] used an averaging technique to study the second-order solutions by introducing his mean elements. Brouwer[7] used the Von Zeipel method to study the same problem. Blitzer[8] introduced dimensionless variables and adopted the method of Bogoliuboff and Mitropolsky[12] to obtain a second-order solution. Kyner[9] used the method of averaging to study the problem with the use of spherical coordinates and angular momentum. In Ref. 10, the authors also applied the method of averaging to obtain the averaged variational equations to the second order. In their analysis, however, the second harmonic of the Earth's gravitational potential was the only perturbation under consideration and the mean anomaly, rather than time t, was treated as the independent variable. In addition, differences are observed when their solutions are compared with those given by Kozai[6] and Brouwer.[7] In Ref. 11, the authors presented their second-order solutions in terms of the mean elements introduced by Kozai, but the analysis was omitted.

In a later work, Kozai[1,3] used the Von Zeipel method to extend Brouwer's theory to include the third-order secular and second-order periodic effects. Recently, Aksnes[14] presented a second-order treatment of the problem through the use of an intermediate orbit[15] and the Hori-Lie perturbation method.[16] More recently, Dallas[17] applied the two-variable asymptotic expansions and Taylor's series expansions to study the perturbing effects to the motion of a satellite due to the second harmonic of the Earth's gravitational potential.

II. Method of Averaging

Consider a system of first-order ordinary differential equations written in the component form†

$$\begin{aligned} \dot{x}_i &= \varepsilon X_{1i}(x_m; y_n) + \varepsilon^2 X_{2i}(x_m; y_n) \\ \dot{y}_j &= Z_j(x_m) + \varepsilon Y_{1j}(x_m; y_n) + \varepsilon^2 Y_{2j}(x_m; y_n) \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, $1 \leq m \leq M$, and $1 \leq n \leq N$ with the initial conditions $x_m(0) = a_m$, $y_n(0) = b_n$. The x_m, are referred to as slow variables because their time variations are proportional to the small parameter ε. The y_n, are referred to as fast variables because the dominant parts of their time variations are proportional to t. The functions $X_{1i}(x_m; y_n)$, $X_{2i}(x_m; y_n)$, $Y_{1j}(x_m; y_n)$, and $Y_{2j}(x_m; y_n)$ are assumed to be continuous functions of x_m and y_n with a period of 2π. In general, the dynamical system, Eqs. (1), is nonlinear in nature and complex in form and hence the integration of the system is usually analytically intractable.

As the first step in the solution, a transformation is introduced

$$\begin{aligned} x_i &= \bar{x}_i + \varepsilon P_{1i}(\bar{x}_m; \bar{y}_n) + \varepsilon^2 P_{2i}(\bar{x}_m; \bar{y}_n) \\ y_j &= \bar{y}_j + \varepsilon Q_{1j}(\bar{x}_m; \bar{y}_n) + \varepsilon^2 Q_{2j}(\bar{x}_m; \bar{y}_n) \end{aligned} \tag{2}$$

so that in a sense the differential equations become simpler to handle in terms of the new variables $\bar{x}_m$ and $\bar{y}_n$. Here $\bar{x}_i$ and $\bar{y}_j$ are regarded as the new unknowns and $P_{1i}(\bar{x}_m; \bar{y}_n)$, $P_{2i}(\bar{x}_m; \bar{y}_n)$, $Q_{1j}(\bar{x}_m; \bar{y}_n)$, $Q_{2j}(\bar{x}_m; \bar{y}_n)$ and also are periodic functions of each y_n as new functions to be determined in such a way as to effect a simplification in the transformed dynamical system. It is desired that the fast variables $\bar{y}_n$ (to the second-order in ε) be eliminated from the transformed differential equations. Thus, the transformed differential equations are to have the form

$$\begin{aligned} \dot{\bar{x}}_i &= \varepsilon U_{1i}(\bar{x}_m) + \varepsilon^2 U_{2i}(\bar{x}_m) + \varepsilon^3 W_{1i}(\bar{x}_m; \bar{y}_n; \varepsilon) \\ \dot{\bar{y}}_j &= Z_j(\bar{x}_m) + \varepsilon V_{1j}(\bar{x}_m) + \varepsilon^2 V_{2j}(\bar{x}_m) + \varepsilon^3 W_{2j}(\bar{x}_m; \bar{y}_n; \varepsilon) \end{aligned} \tag{3}$$

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† If F_i, $i = 1, 2, \ldots, L$, are functions of the variables x_m, $m = 1, 2, \ldots, M$, we write $F_i(x_m)$. If F_i are functions of variables x_m and y_n, $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$, we write $F_i(x_m; y_n)$.

for suitable functions U_{1i}, U_{2i}, V_{1j}, V_{2j}, W_{1i}, and W_{2j}. For a second-order theory, the third-order terms W_{1i} and W_{2j} will be ignored. Thus Eqs. (3) become

$$\dot{\bar{x}}_i = \varepsilon U_{1i}(\bar{x}_m) + \varepsilon^2 U_{2i}(\bar{x}_m), \quad \dot{\bar{y}}_j = Z_j(\bar{x}_m) + \varepsilon V_{1j}(\bar{x}_m) + \varepsilon^2 V_{2j}(\bar{x}_m) \tag{4}$$

with initial conditions $\bar{x}_m(0) = \bar{a}_m$, $\bar{y}_m(0) = \bar{b}_n$. These initial values are obtained by substituting the initial values a_m, b_n into the transformation (2). The explicit expressions of functions U_{1i}, U_{2i}, V_{1j}, V_{2j}, P_{1i}, P_{2i}, Q_{1j}, and Q_{2j} will be given here without proof. For the thorough discussions of the general theory of the method of averaging, see Refs. 4, 12, and 18.

To describe the explicit expressions of these functions, the necessary relations and their definitions will be introduced. To begin with, the Fourier series expression for the perturbing function X_{1i} has the form

$$X_{1i}(\bar{x}_m; \bar{y}_n) = X_{1i0}(\bar{x}_m) + X_{1i1}(\bar{x}_m; \bar{y}_n) \tag{5}$$

where

$$X_{1i0}(\bar{x}_m) = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \cdots \int_0^{2\pi} X_{1i}(\bar{x}_m; \bar{y}_n)\, d\bar{y}_1 \ldots d\bar{y}_N$$

$$X_{1i1}(\bar{x}_m; \bar{y}_n) = \sum_{\mathbf{k}} \{X_{1i1\mathbf{k}c}(\bar{x}_m)\cos[\mathbf{k}, \bar{y}] + X_{1i1\mathbf{k}s}(\bar{x}_m)\sin[\mathbf{k}, \bar{y}]\} \tag{6}$$

$$[\mathbf{k}, \bar{y}] = \sum_{n=1}^{N} k_n \bar{y}_n$$

In Eqs. (6), the following definitions apply. The notation $\mathbf{k} = (k_1, k_2, \ldots, k_n, \ldots, k_N)$ is used to denote a vector, each component k_n is an integer; $\mathbf{k} = \mathbf{0}$ is not permitted. The notation $\sum_{\mathbf{k}}$ indicates summation over all possible integer vectors $\mathbf{k}$. The notations $X_{1i1kc}(\bar{x}_m)$ and $X_{1i1ks}(\bar{x}_m)$ denote the coefficients of the $\cos[\mathbf{k}, \bar{y}]$ and $\sin[\mathbf{k}, \bar{y}]$ terms, respectively, in the summation. These Fourier-series coefficients are to be determined in the usual way. It can be shown by the method of averaging that

$$U_{1i}(\bar{x}_m) = X_{1i0}(\bar{x}_m), \qquad V_{1j}(\bar{x}_m) = Y_{1j0}(\bar{x}_m) \tag{7}$$

and to within an arbitrary function‡ of $\bar{x}_m$, that

$$P_{1i}(\bar{x}_m; \bar{y}_n) = -\sum_{\mathbf{k}} [\mathbf{k}, Z(\bar{x}_m)]^{-1} \times \{X_{1i1\mathbf{k}s}(\bar{x}_m)\cos[\mathbf{k}, \bar{y}] - X_{1i1\mathbf{k}c}(\bar{x}_m)\sin[\mathbf{k}, \bar{y}]\}$$
$$Q_{1j}(\bar{x}_m; \bar{y}_n) = -\sum_{\mathbf{k}} [\mathbf{k}, Z(\bar{x}_m)]^{-1} \times \{S_{1j\mathbf{k}s}(\bar{x}_m)\cos[\mathbf{k}, \bar{y}] - S_{1j\mathbf{k}c}(\bar{x}_m)\sin[\mathbf{k}, \bar{y}]\} \tag{8}$$

where

$$S_{1j} = \sum_{r=1}^{M} P_{1r}(\bar{x}_m; \bar{y}_n)\frac{\partial Z_j(\bar{x}_m)}{\partial \bar{x}_r} + Y_{1j1}(\bar{x}_m; \bar{y}_n) \tag{9}$$

In Eqs. (7) and (8), the notations $Y_{1j0}(\bar{x}_m)$, $Y_{1j1}(\bar{x}_m; \bar{y}_n)$, $S_{1j\mathbf{k}s}(\bar{x}_m)$ and $S_{1j\mathbf{k}c}(\bar{x}_m)$ have the similar definitions which apply for $X_{1i0}(\bar{x}_m)$, $X_{1i1}(\bar{x}_m; \bar{y}_n)$, $X_{1i1\mathbf{k}s}(\bar{x}_m)$ and $X_{1i1\mathbf{k}c}(\bar{x}_m)$, respectively. Secondly, two new functions are introduced

$$R_{1i}(\bar{x}_m; \bar{y}_n) = X_{2i}(\bar{x}_m; \bar{y}_n) + \sum_{r=1}^{M}\left[P_{1r}(\bar{x}_m; \bar{y}_n)\frac{\partial X_{1i}(\bar{x}_m; \bar{y}_n)}{\partial \bar{x}_r} - U_{1r}(\bar{x}_m)\frac{\partial P_{1i}(\bar{x}_m; \bar{y}_n)}{\partial \bar{x}_r}\right] + \sum_{s=1}^{N}\left[Q_{1s}(\bar{x}_m; \bar{y}_n)\frac{\partial X_{1i}(\bar{x}_m; \bar{y}_n)}{\partial \bar{y}_s} - V_{1s}(\bar{x}_m)\frac{\partial P_{1i}(\bar{x}_m; \bar{y}_n)}{\partial \bar{y}_s}\right]$$
$$R_{2j}(\bar{x}_m; \bar{y}_n) = Y_{2j}(\bar{x}_m; \bar{y}_n) + \frac{1}{2}\sum_{r=1}^{M}\sum_{w=1}^{M} P_{1r}(\bar{x}_m; \bar{y}_n)P_{1w}(\bar{x}_m; \bar{y}_n)\frac{\partial^2 Z_j(\bar{x}_m)}{\partial \bar{x}_r \partial \bar{x}_w} + \sum_{r=1}^{M}\left[P_{1r}(\bar{x}_m; \bar{y}_n)\frac{\partial Y_{1j}(\bar{x}_m; \bar{y}_n)}{\partial \bar{x}_r} - U_{1r}(\bar{x}_m)\frac{\partial Q_{1j}(\bar{x}_m; \bar{y}_n)}{\partial \bar{x}_r}\right] + \sum_{s=1}^{N}\left[Q_{1s}(\bar{x}_m; \bar{y}_n)\frac{\partial Y_{1j}(\bar{x}_m; \bar{y}_n)}{\partial \bar{y}_s} - V_{1s}(\bar{x}_m)\frac{\partial Q_{1j}(\bar{x}_m; \bar{y}_n)}{\partial \bar{y}_s}\right] \tag{10}$$

‡ For the discussions of the arbitrariness, see Ref. 4.

The method of averaging shows that

$$U_{2i}(\bar{x}_m) = R_{1i0}(\bar{x}_m), \quad V_{2j}(\bar{x}_m) = R_{2j0}(\bar{x}_m) \tag{11}$$

and to within an arbitrary function of $\bar{x}_m$, that

$$P_{2i}(\bar{x}_m; \bar{y}_n) = -\sum_{\mathbf{k}} [\mathbf{k}, Z(x_m)]^{-1} \times \{R_{1i1\mathbf{k}s}(\bar{x}_m)\cos[\mathbf{k}, \bar{y}] - R_{1i1\mathbf{k}c}(\bar{x}_m)\sin[\mathbf{k}, \bar{y}]\}$$
$$Q_{2j}(\bar{x}_m; \bar{y}_n) = -\sum_{\mathbf{k}} [\mathbf{k}, Z(\bar{x}_m)]^{-1} \times \{S_{2j\mathbf{k}s}(\bar{x}_m)\cos[\mathbf{k}, \bar{y}] - S_{2j\mathbf{k}c}(\bar{x}_m)\sin[\mathbf{k}, \bar{y}]\} \tag{12}$$

where

$$S_{2j} = \sum_{r=1}^{M} P_{2r}(\bar{x}_m; \bar{y}_n)\frac{\partial Z_j(\bar{x}_m)}{\partial \bar{x}_r} + R_{2j1}(\bar{x}_m; \bar{y}_n) \tag{13}$$

Again the notations $R_{1i0}(\bar{x}_m)$, $R_{1i1}(\bar{x}_m; \bar{y}_n)$, $R_{1i1\mathbf{k}s}(\bar{x}_m)$, $R_{1i1\mathbf{k}c}(\bar{x}_m)$; $R_{2j0}(\bar{x}_m)$, $R_{2j1}(\bar{x}_m; \bar{y}_n)$, and $S_{2j\mathbf{k}s}$, $S_{2j\mathbf{k}c}(\bar{x}_m)$ have the usual meaning in the Fourier series expressions for R_{1i}, R_{2j} and S_{2j}, respectively. It should also be noted, in arriving at these expressions, that the nonresonant condition, i.e., $[\mathbf{k}, Z(\bar{x}_m)] \neq 0$, has been assumed.

Following the determination of the functions U's, V's, P's, and Q's, Eqs. (4) may either be integrated analytically or integrated numerically with a much longer integration step time than may be used with Eqs. (1). The second-order solution for x_i and y_j can then be obtained from Eqs. (2).

III. Equations of Motion

The results arrived at in this section are based on several assumptions. It is assumed that the satellite moves in an elliptic orbit about an axially symmetric Earth. The satellite itself is assumed to be a point mass. It is further assumed that the kinetic energy of the satellite is large compared to the effects of the perturbing gravitational potential. It is also understood that the Earth's gravitational potential is assumed to be the only perturbing factor under consideration.

Although the averaged differential equations associated with the semimajor axis a, eccentricity e, inclination i, ascending node Ω, argument of perigee ω, and mean anomaly M can be obtained in a straightforward manner by following the procedures outlined in Sec. II, the Delaunay variables will be used because of the simplicity shown by Lorell et al.[10]

The Delaunay variables are defined as follows:

$$\begin{aligned} L_1 &= L = (\mu a)^{1/2} & l_1 &= l = M \\ L_2 &= G = L(1-e^2)^{1/2} & l_2 &= g = \omega \\ L_3 &= H = G\cos i & l_3 &= h = \Omega \end{aligned} \tag{14}$$

where μ is the product of the gravitational constant and the mass of the Earth.

If a small parameter ε is chosen to be of the order of 10^{-3}, the associated equations of motion have the form

$$\begin{aligned} \dot{L}_i &= \varepsilon\frac{\partial E_1}{\partial l_i} + \varepsilon^2\frac{\partial E_2}{\partial l_i}, \qquad i = 1,2,3 \\ \dot{l}_i &= -\varepsilon\frac{\partial E_1}{\partial L_i} - \varepsilon^2\frac{\partial E_2}{\partial L_i}, \qquad i = 2,3 \\ \dot{l}_1 &= \frac{\mu^2}{L_1{}^3} - \varepsilon\frac{\partial E_1}{\partial L_1} - \varepsilon^2\frac{\partial E_2}{\partial L_1} \end{aligned} \tag{15}$$

where εE_1 stands for the portion of the perturbing Hamiltonian associated with the second harmonic which contains the small coefficient J_2, and $\varepsilon^2 E_2$ stands for the one associated with the third and fourth harmonics which contain the small coefficient J_3 and J_4, respectively. Explicitly we have

$$E_1 = \left(\frac{J_2}{\varepsilon}\right)\frac{\mu^4 R^2}{2L_1{}^6}\left(\frac{a}{r}\right)^3 [A + B\cos 2(g+f)]$$

$$E_2 = -\left(\frac{J_3}{\varepsilon^2}\right)\frac{\mu^5 R^3}{2L_1{}^8}\left(\frac{a}{r}\right)^4\left(\frac{2}{3}B\right)^{1/2}\left[\left(\frac{5}{2}B-3\right)\sin(g+f) - \frac{5}{6}B\sin 3(g+f)\right] - \frac{35}{8}\left(\frac{J_4}{\varepsilon^2}\right)\frac{\mu^6 R^4}{L_1{}^{10}}\left(\frac{a}{r}\right)^5 \times \left[\frac{3}{35} - \frac{2}{7}B + \frac{1}{6}B^2 + \frac{2}{3}B\left(\frac{3}{7} - \frac{B}{3}\right)\cos 2(g+f) + \frac{B^2}{18}\cos 4(g+f)\right] \tag{16}$$

where

$$A = -\frac{1}{2} + \frac{3}{2}\left(\frac{H}{G}\right)^2, \qquad B = \frac{3}{2}\left(1 - \frac{H^2}{G^2}\right)$$

and where r, R, and f designate the magnitude of the position vector of the satellite with respect to the geocenter, the equatorial radius of the earth and the true anomaly. Here variables L_i, $i = 1,2,3$, l_2, and l_3 are regarded as slow variables and l_1 is regarded as the fast variable.

It can be readily shown that the partial derivatives of E_1, E_2 with respect to the variables L_i, l_i, $i = 1,2,3$, are continuous functions of L_i and l_i, $i = 1,2,3$; and also are periodic functions of l_1 with a period of 2π. Thus the method of averaging outlined in the preceding section can be applied to the dynamical system (15).

IV. Solutions

The averaging procedure used here is viewed as being composed of three steps. In the first step, a transformation of variables is introduced to replace the dynamical system with an intermediate set of averaged differential equations in which (to second-order in a small parameter) the fast variable M (mean anomaly) with a 0th-order rate has been eliminated. In the second step, another transformation of variables is introduced to replace the first intermediate dynamical system with a second intermediate set of secular (or doubly averaged) differential equations in which the fast variable ω (argument of perigee) with a first-order rate has been eliminated. In the third step, the secular differential equations are integrated, either numerically or (if possible) analytically, to obtain second-order, secular solutions.

Second-Order Averaged Differential Equations

If we let x_1, x_2, x_3, x_4, x_5, and y denote the orbital elements L_1, L_2, L_3, l_2, l_3, and l_1, respectively, Eqs. (2) and (4) ($M = 5$, $N = 1$) would provide the proper transformation from the osculating elements L_i, l_i, $i = 1,2,3$, to the averaged elements $\bar{L}_i$, $\bar{l}_i$, $i = 1,2,3$, and the transformed differential equations, respectively, for the dynamical system (15).

For simplicity, in the remainder of the section, all the variables without a bar are understood to have their averaged values. Making use of some well-known relations[19] in orbital mechanics and the relations

$$\dot{a} = \frac{2}{\mu}L\dot{L} \qquad \dot{\omega} = \dot{g}$$

$$\dot{e} = \frac{1}{eL}\left[(1-e^2)\dot{L} - (1-e^2)^{1/2}\dot{G}\right] \qquad \dot{\Omega} = \dot{h} \tag{17}$$

$$\dot{(i)} = \frac{1}{G\sin i}(\cos i\,\dot{G} - \dot{H}) \qquad \dot{M} = \dot{l}$$

The averaged, or transformed, dynamical system, after straightforward manipulation, takes the form

$$\dot{a} = 0$$

$$\dot{e} = -\frac{3}{32}nJ_2{}^2\left(\frac{R}{p}\right)^4 \sin^2 i(14-15\sin^2 i)e(1-e^2)\sin 2\omega - \frac{3}{8}nJ_3\left(\frac{R}{p}\right)^3 \sin i(4-5\sin^2 i)(1-e^2)\cos\omega - \frac{15}{32}nJ_4\left(\frac{R}{p}\right)^4 \sin^2 i(6-7\sin^2 i)e(1-e^2)\sin 2\omega$$

$$\dot{(i)} = \frac{3}{64}nJ_2{}^2\left(\frac{R}{p}\right)^4 \sin 2i(14-15\sin^2 i)e^2\sin 2\omega + \frac{3}{8}nJ_3\left(\frac{R}{p}\right)^3 \cos i(4-5\sin^2 i)e\cos\omega + \frac{15}{64}nJ_4\left(\frac{R}{p}\right)^4 \sin 2i(6-7\sin^2 i)e^2\sin 2\omega$$

$$\dot{\omega} = \frac{3}{4}nJ_2\left(\frac{R}{p}\right)^2(4-5\sin^2 i) + \frac{3}{16}nJ_2{}^2\left(\frac{R}{p}\right)^4 \times \left\{48 - 103\sin^2 i + \frac{215}{4}\sin^4 i + \left(7 - \frac{9}{2}\sin^2 i - \frac{45}{8}\sin^4 i\right)e^2 + 6\left(1 - \frac{3}{2}\sin^2 i\right)(4-5\sin^2 i)(1-e^2)^{1/2} - \frac{1}{4}\left[2(14-15\sin^2 i)\sin^2 i - (28-158\sin^2 i+135\sin^4 i)e^2\right] \times \cos 2\omega\right\} + \frac{3}{8}nJ_3\left(\frac{R}{p}\right)^3\left[(4-5\sin^2 i)\frac{\sin^2 i - e^2\cos^2 i}{e\sin i} + 2\sin i(13-15\sin^2 i)e\right]\sin\omega - \frac{15}{32}nJ_4\left(\frac{R}{p}\right)^4 \times \left\{16 - 62\sin^2 i + 49\sin^4 i + \frac{3}{4}(24-84\sin^2 i+63\sin^4 i)e^2 + \left[\sin^2 i(6-7\sin^2 i) - \frac{1}{2}(12-70\sin^2 i+63\sin^4 i)e^2\right]\cos 2\omega\right\}$$

$$\dot{\Omega} = -\frac{3}{2}nJ_2\left(\frac{R}{p}\right)^2\cos i - \frac{3}{2}nJ_2{}^2\left(\frac{R}{p}\right)^4\cos i\left\{\frac{9}{4} + \frac{3}{2}(1-e^2)^{1/2} - \sin^2 i\left[\frac{5}{2} + \frac{9}{4}(1-e^2)^{1/2}\right] + \frac{e^2}{4}\left(1 + \frac{5}{4}\sin^2 i\right) + \frac{e^2}{8}(7-15\sin^2 i)\cos 2\omega\right\} - \frac{3}{8}nJ_3\left(\frac{R}{p}\right)^3 \times (15\sin^2 i - 4)e\cot i\sin\omega + \frac{15}{16}nJ_4\left(\frac{R}{p}\right)^4\cos i \times \left[(4-7\sin^2 i)\left(1 + \frac{3}{2}e^2\right) - (3-7\sin^2 i)e^2\cos 2\omega\right]$$

$$\dot{M} = n\left[1 + \frac{3}{2}J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)(1-e^2)^{1/2}\right] + \frac{3}{2}nJ_2{}^2\left(\frac{R}{p}\right)^4\left\{\left(1 - \frac{3}{2}\sin^2 i\right)^2(1-e^2) + \left[\frac{5}{4}\left(1 - \frac{5}{2}\sin^2 i + \frac{13}{8}\sin^4 i\right) + \frac{5}{8}\left(1 - \sin^2 i + \frac{5}{8}\sin^4 i\right)e^2 + \frac{1}{16}\sin^2 i(14-15\sin^2 i)\left(1 - \frac{5}{2}e^2\right)\cos 2\omega\right](1-e^2)^{1/2}\right\} + \frac{3}{8}nJ_2{}^2\left(\frac{R}{p}\right)^4(1-e^2)^{1/2}\left\{3\left[3 - \frac{15}{2}\sin^2 i + \frac{47}{8}\sin^4 i + \left(\frac{3}{2} - 5\sin^2 i + \frac{117}{6}\sin^4 i\right)e^2 - \frac{1}{8}\left(1 + 5\sin^2 i - \frac{101}{8}\sin^4 i\right)e^4\right] + \frac{e^2}{8}\sin^2 i \times \left[70 - 123\sin^2 i + (56-66\sin^2 i)e^2\right]\cos 2\omega + \frac{27}{128}e^4\sin^4 i\cos 4\omega\right\} - \frac{3}{8}nJ_3\left(\frac{R}{p}\right)^3\sin i(4-5\sin^2 i) \times \frac{1-4e^2}{e}(1-e^2)^{1/2}\sin\omega + \frac{15}{64}nJ_4\left(\frac{R}{p}\right)^4 \times \sin^2 i(6-7\sin^2 i)(2-5e^2)(1-e^2)^{1/2}\cos 2\omega \tag{18}$$

with initial conditions a_o, e_o, i_o, Ω_o, ω_o, M_o at $t = t_o$.

Short-Periodic Variations

At various points in the previous discussion, explicit expressions were derived for the differential equations of the averaged elements without having derived the short-periodic variations explicitly. The results derived thus far can serve as a solid basis to estimate the long-term motion of a satellite under the influence of earth gravitational potential. These results cannot, however, be used to generate accurate trajectory data without adding the effects contributed by the short-periodic variations. For this reason, the explicit expressions will now be presented for the first-order, short-periodic variation associated with each of the orbital elements. Second-order, short-periodic variations will not be included because of their magnitude and their periodicity.

From Eqs. (8), for a dynamical system with one fast variable, l, it can be shown to within an arbitrary function of the slow variables, that

$$\varepsilon P_{1a} = \frac{2L^4}{\mu^3}\frac{\partial \tilde{E}_{11}}{\partial l} \qquad \varepsilon P_{1\omega} = -\frac{1}{n}\frac{\partial \tilde{E}_{11}}{\partial G}$$

$$\varepsilon P_{1e} = \frac{1}{n}\left[\frac{1-e^2}{eL}\frac{\partial \tilde{E}_{11}}{\partial l} - \frac{(1-e^2)^{1/2}}{eL}\frac{\partial \tilde{E}_{11}}{\partial g}\right] \qquad \varepsilon P_{1\Omega} = -\frac{1}{n}\frac{\partial \tilde{E}_{11}}{\partial H}$$

$$\varepsilon P_{1i} = \frac{1}{n}\left[\frac{\cot i}{G}\frac{\partial \tilde{E}_{11}}{\partial g}\right] \qquad \varepsilon Q_{1M} = -\frac{\partial}{\partial L}\left(\frac{\tilde{E}_{11}}{n}\right) \tag{19}$$

where

$$\tilde{E}_{11} = \frac{\mu^4 J_2 R^2}{2L^3G^3}\left[A(f-l+e\sin f) + \frac{B}{2}e\sin(2g+f) + \frac{B}{2}\sin 2(g+f) + \frac{B}{6}e\sin(2g+3f)\right] \tag{20}$$

Here the second sub-index designates the associated orbital element. Their explicit expressions are

$$\varepsilon P_{1a} = J_2\left(\frac{R^2}{a}\right)\left\{\left(\frac{a}{r}\right)^3\left[\left(1-\frac{3}{2}\sin^2 i\right)+\frac{3}{2}\sin^2 i\cos 2(\omega+f)\right] - \left(1-\frac{3}{2}\sin^2 i\right)(1-e^2)^{-3/2}\right\}$$

$$\begin{aligned}\varepsilon P_{1e} = {} & \frac{1}{2}J_2\left(\frac{R}{p}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{1}{e}\left[1+\frac{3}{2}e^2-(1-e^2)^{3/2}\right] + 3\left(1+\frac{e^2}{4}\right)\cos f + \frac{3}{2}e\cos 2f + \frac{e^2}{4}\cos 3f\right\} + \\ & \frac{3}{8}J_2\left(\frac{R}{p}\right)^2\sin^2 i\left[\left(1+\frac{11}{4}e^2\right)\cos(2\omega+f) + \frac{e^2}{4}\cos(2\omega-f) + 5e\cos(2\omega+2f) + \frac{1}{3}\left(7+\frac{17}{4}e^2\right)\times \right.\\ & \left.\cos(2\omega+3f) + \frac{3}{2}e\cos(2\omega+4f) + \frac{e^2}{4}\cos(2\omega+5f) + \frac{3}{2}e\cos 2\omega\right]\end{aligned}$$

$$\varepsilon P_{1i} = \frac{3}{8}J_2\left(\frac{R}{p}\right)^2\sin 2i \times \left[e\cos(2\omega+f) + \cos 2(\omega+f) + \frac{e}{3}\cos(2\omega+3f)\right]$$

$$\begin{aligned}\varepsilon P_{1\omega} = {} & \frac{3}{4}J_2\left(\frac{R}{p}\right)^2(4-5\sin^2 i)(f-M+e\sin f) + \\ & \frac{3}{2}J_2\left(\frac{R}{p}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)\left[\frac{1}{e}\left(1-\frac{1}{4}e^2\right)\sin f + \frac{1}{2}\sin 2f + \frac{1}{12}e\sin 3f\right] - \frac{3}{2}J_2\left(\frac{R}{p}\right)^2\left\{\frac{1}{e}\left[\frac{1}{4}\sin^2 i + \right.\right.\\ & \left.\frac{e^2}{2}\left(1-\frac{15}{8}\sin^2 i\right)\right]\sin(2\omega+f) + \frac{e}{16}\sin^2 i\sin(2\omega-f) + \\ & \frac{1}{2}\left(1-\frac{5}{2}\sin^2 i\right)\sin 2(\omega+f) - \frac{1}{e}\left[\frac{7}{12}\sin^2 i - \frac{e^2}{6}\left(1-\frac{19}{8}\sin^2 i\right)\right]\sin(2\omega+3f) - \\ & \left.\frac{3}{8}\sin^2 i\sin(2\omega+4f) - \frac{1}{16}e\sin^2 i\sin(2\omega+5f)\right\} - \frac{9}{16}J_2\left(\frac{R}{p}\right)^2\sin^2 i\sin 2\omega\end{aligned}$$

$$\varepsilon P_{1\Omega} = -\frac{3}{2}J_2\left(\frac{R}{p}\right)^2\cos i\left[f-M+e\sin f - \frac{e}{2}\sin(2\omega+f) - \frac{1}{2}\sin 2(\omega+f) - \frac{e}{6}\sin(2\omega+3f)\right]$$

$$\begin{aligned}\varepsilon Q_{1M} = {} & -\frac{3}{2}J_2\left(\frac{R}{p}\right)^2\frac{(1-e^2)^{1/2}}{e}\left\{\left(1-\frac{3}{2}\sin^2 i\right)\times \left[\left(1-\frac{1}{4}e^2\right)\sin f + \frac{e}{2}\sin 2f + \frac{e^2}{12}\sin 3f\right] + \right.\\ & \frac{1}{2}\sin^2 i\left[-\frac{1}{2}\left(1+\frac{5}{4}e^2\right)\sin(2\omega+f) - \frac{e^2}{8}\sin(2\omega-f) + \frac{7}{6}\left(1-\frac{e^2}{28}\right)\sin(2\omega+3f) + \frac{3}{4}e\sin(2\omega+4f) + \right.\\ & \left.\left.\frac{e^2}{8}\sin(2\omega+5f)\right]\right\} + \frac{9}{16}J_2\left(\frac{R}{p}\right)^2(1-e^2)^{1/2}\sin^2 i\sin 2\omega\end{aligned} \tag{21}$$

Secular and the Long-Periodic Variation

It is revealed, from Eqs. (18), that the right-hand sides of these equations are continuous functions of orbital elements, a, e, i, and ω and that they are periodic functions of ω with period 2π. It is also noted that ω is the only variable appearing in the right-hand sides of these equations which has a secular variation of first order. In this subsection, the technique of the method of averaging is applied once more to determine the periodic variations associated with argument of perigee and the secular rates of the six orbital elements. Since the period of these periodic variations is of the order $1/\varepsilon$, they will be referred to as the long-periodic perturbing terms in order to distinguish them from the short-periodic terms associated with mean anomaly and having a period of the order of one.

Regarding the elements a, e, i as the new slow variables, $\tilde{x}_j$, $j=1,2,3$, and elements ω, Ω, and M as the new fast variables, $\tilde{y}_j$, $j=1,2,3$, Eqs. (18) have the form

$$\begin{aligned}\dot{\tilde{x}}_j &= \varepsilon\tilde{X}_j(\tilde{x}_k;\tilde{y}_1) \qquad k,j=1,2,3\\ \dot{\tilde{y}}_j &= \tilde{Z}_j(\tilde{x}_k)+\varepsilon\tilde{Y}_j(\tilde{x}_k;\tilde{y}_1)\end{aligned} \tag{22}$$

where $\tilde{X}_2$, $\tilde{X}_3$, $\tilde{Y}_1$, $\tilde{Y}_2$, and $\tilde{Y}_3$ are the second-order, averaged perturbing functions for e, i, ω, Ω, and M, respectively, divided by the small parameter ε, $\dot{\tilde{x}}_1 = 0$, and

$$\begin{aligned}\tilde{Z}_1 &= \frac{3}{4}nJ_2\left(\frac{R}{p}\right)^2(4-5\sin^2 i)\\ \tilde{Z}_2 &= -\frac{3}{2}nJ_2\left(\frac{R}{p}\right)^2\cos i\\ \tilde{Z}_3 &= n\left[1+\frac{3}{2}J_2\left(\frac{R}{p}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)(1-e^2)^{1/2}\right]\end{aligned} \tag{23}$$

Now as before, a second transformation is introduced

$$\begin{aligned}\tilde{x}_j &= x_{js}+\varepsilon x_{jL}(x_{ks};y_{1s}) \qquad k,j=1,2,3\\ \tilde{y}_j &= y_{js}+\varepsilon y_{jL}(x_{ks};y_{1s})\end{aligned} \tag{24}$$

such that the transformed (doubly-averaged) system is to be of the form

$$\begin{aligned}\dot{x}_{js} &= \varepsilon\tilde{U}_j(x_{ks}) \qquad k,j=1,2,3\\ \dot{y}_{js} &= \tilde{Z}_j(x_{ks})+\varepsilon\tilde{V}_j(x_{ks})\end{aligned} \tag{25}$$

for suitable functions x_{jL}, y_{jL}, $\tilde{U}_j$, and $\tilde{V}_j$. It is noted that the subscripts s and L are used to designate the secular (doubly-averaged) elements and the long-periodic variations, respectively.

In Eqs. (24), the second-order long-periodic variations are ignored since the third-order perturbing effects have not been included in this paper. These functions can be determined by Eqs. (7) and (8). They are

$$\dot{a}_s = \dot{e}_s = \dot{i}_s = 0$$

$$\begin{aligned}\dot{\omega}_s = {} & \frac{3}{4} n J_2 \left(\frac{R}{p_s}\right)^2 (4 - 5\sin^2 i_s) + \frac{3}{4} n J_2^2 \left(\frac{R}{p_s}\right)^4 \times \\ & \left[12 - \frac{103}{4}\sin^2 i_s + \frac{215}{16}\sin^4 i_s + \right. \\ & \left(\frac{7}{4} - \frac{9}{8}\sin^2 i_s - \frac{45}{32}\sin^4 i_s\right) e_s^2 + \frac{3}{2}\left(1 - \frac{3}{2}\sin^2 i_s\right) \times \\ & \left.(4 - 5\sin^2 i_s)(1 - e_s^2)^{1/2}\right] - \frac{15}{32} n J_4 \left(\frac{R}{p_s}\right)^4 \times \\ & \left[16 - 62\sin^2 i_s + 49\sin^4 i_s + \frac{3}{4}(24 - 84\sin^2 i_s + 63\sin^4 i_s) e_s^2\right]\end{aligned}$$

$$\begin{aligned}\dot{\Omega}_s = {} & -\frac{3}{2} n J_2 \left(\frac{R}{p_s}\right)^2 \cos i_s - \frac{3}{2} n J_2^2 \left(\frac{R}{p_s}\right)^4 \times \\ & \cos i_s \left\{\frac{9}{4} + \frac{3}{2}(1 - e_s^2)^{1/2} - \sin^2 i_s \left[\frac{5}{2} + \frac{9}{4}(1 - e_s^2)^{1/2}\right] + \right. \\ & \left.\frac{e^2}{4}\left(1 + \frac{5}{4}\sin^2 i_s\right)\right\} + \frac{15}{16} n J_4 \left(\frac{R}{p_s}\right)^4 \times \\ & (4 - 7\sin^2 i_s)\cos i_s \left(1 + \frac{3}{2} e_s^2\right)\end{aligned}$$

$$\begin{aligned}\dot{M}_s = {} & n\left[1 + \frac{3}{2} J_2 \left(\frac{R}{p_s}\right)^2 \left(1 - \frac{3}{2}\sin^2 i_s\right)(1 - e_s^2)^{1/2}\right] + \\ & \frac{15}{16} n J_2^2 \left(\frac{R}{p_s}\right)^4 (1 - e_s^2)^{1/2} \left[2 - 5\sin^2 i_s + \frac{13}{4}\sin^4 i_s + \right. \\ & \left(1 - \sin^2 i_s - \frac{5}{8}\sin^4 i_s\right) e_s^2 + \frac{8}{5}\left(1 - \frac{3}{2}\sin^2 i_s\right)^2 \times \\ & \left.(1 - e_s^2)^{1/2}\right] + \frac{9}{8} n J_2^2 \left(\frac{R}{p_s}\right)^4 (1 - e_s^2)^{-1/2} \times \\ & \left[3 - \frac{15}{2}\sin^2 i_s + \frac{47}{8}\sin^4 i_s + \left(\frac{3}{2} - 5\sin^2 i_s + \frac{117}{16}\sin^4 i_s\right) e_s^2 - \right. \\ & \left.\frac{1}{8}\left(1 + 5\sin^2 i_s - \frac{101}{8}\sin^4 i_s\right) e_s^4\right] - \frac{45}{128} n J_4 \left(\frac{R}{p_s}\right)^4 \times \\ & (8 - 40\sin^2 i_s + 35\sin^4 i_s) e_s^2 (1 - e_s^2)^{1/2} \qquad (26)\end{aligned}$$

and where $P_s = a(1 - e_s^2)$. The long-periodic variations are

$$\begin{aligned}\varepsilon e_L = {} & \left[\frac{1}{16} J_2 \left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s (14 - 15\sin^2 i_s) e_s \cos 2\omega_s - \right. \\ & \frac{1}{2}\left(\frac{J_3}{J_2}\right)\left(\frac{R}{p_s}\right) \sin i_s \sin\omega_s + \frac{5}{16}\frac{J_4}{J_2}\left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s (6 - 7\sin^2 i_s) \times \\ & \left. e_s^2 \cos 2\omega_s \right](1 - e_s^2)\end{aligned}$$

$$\begin{aligned}\varepsilon i_L = {} & -\frac{1}{32} J_2 \left(\frac{R}{p_s}\right)^2 \gamma \sin 2i_s (14 - 15\sin^2 i_s) e_s^2 \cos 2\omega + \\ & \frac{1}{2}\frac{J_3}{J_2}\left(\frac{R}{p_s}\right) \cos i_s e_s \sin\omega_s - \frac{5}{32}\frac{J_4}{J_2}\left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s \times \\ & (6 - 7\sin^2 i_s) e_s^2 \cos 2\omega_s\end{aligned}$$

$$\begin{aligned}\varepsilon\Omega_L = {} & -\frac{5}{16} J_2 \left(\frac{R}{p_s}\right)^2 \gamma e_s^2 \cos i_s \left[\frac{2}{5}(7 - 15\sin^2 i_s) + \right. \\ & \left.\gamma \sin^2 i_s (14 - 15\sin^2 i_s)\right] \sin 2\omega_s - \frac{1}{2}\frac{J_3}{J_2}\left(\frac{R}{p_s}\right) e_s \cot i_s \cos\omega_s - \\ & \frac{25}{16}\frac{J_4}{J_2}\left(\frac{R}{p_s}\right)^2 \gamma e_s^2 \cos i_s \left[\frac{2}{5}(3 - 7\sin^2 i_s) + \right. \\ & \left.\gamma \sin^2 i_s (6 - 7\sin^2 i_s)\right] \sin 2\omega_s\end{aligned}$$

$$\begin{aligned}\varepsilon\omega_L = {} & -\frac{1}{32} J_2 \left(\frac{R}{p_s}\right)^2 \gamma \{2\sin^2 i_s (14 - 15\sin^2 i_s) \times \\ & [1 - \gamma(13 - 15\sin^2 i_s) e_s^2] - (28 - 158\sin^2 i_s + \\ & 135\sin^4 i_s) e_s^2\} \sin 2\omega_s - \frac{1}{2}\frac{J_3}{J_2}\left(\frac{R}{p_s}\right) \frac{\sin^2 i_s - e_s^2 \cos^2 i_s}{e_s \sin i_s} \cos\omega_s - \\ & \frac{5}{32}\frac{J_4}{J_2}\left(\frac{R}{p}\right)^2 \gamma \{2\sin^2 i_s (6 - 7\sin^2 i_s)[1 - \gamma(13 - 15\sin^2 i_s) e_s^2] - \\ & (12 - 70\sin^2 i_s + 63\sin^4 i_s) e_s^2\} \sin 2\omega_s\end{aligned}$$

$$\begin{aligned}\varepsilon M_L = {} & \frac{1}{16} J_2 \left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s (14 - 15\sin^2 i_s)(1 - e_s^2)^{3/2} \sin 2\omega_s + \\ & \frac{1}{32} J_2 \left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s [(70 - 123\sin^2 i_s) e_s^2 + \\ & 2(28 - 33\sin^2 i_s) e_s^4](1 - e_s^2)^{-1/2} \sin 2\omega_s + \\ & \frac{27}{1024} J_2 \left(\frac{R}{p_s}\right)^2 \sin^4 i_s e_s^4 (1 - e_s^2)^{-1/2} \sin 4\omega_s + \\ & \frac{1}{2}\frac{J_3}{J_2}\left(\frac{R}{p_s}\right) \sin i_s \frac{(1 - e_s^2)}{e_s} \cos\omega_s + \frac{5}{16}\frac{J_4}{J_2}\left(\frac{R}{p_s}\right)^2 \gamma \sin^2 i_s \times \\ & (6 - 7\sin^2 i_s)(1 - e_s^2)^{3/2} \sin 2\omega_s \qquad (27)\end{aligned}$$

where $\gamma = (4 - 5\sin^2 i_s)^{-1}$. The initial conditions for these doubly averaged orbital elements, a_{so}, e_{so}, i_{so}, ω_{so}, Ω_{so}, M_{so}, can be evaluated by transformation (24) and the initial values for the averaged orbital elements a_o, e_o, i_o, ω_o, Ω_o, and M_o.

Transformations and Initializations

The analysis and results discussed in the previous subsections can be used as the basis to study the long-term variations (lifetime study) as well as the short-term variations (trajectory study) for an artificial Earth satellite under the influence of the oblateness of Earth, provided the proper transformation and initialization procedures are used.

The transformation relating the osculating elements x_i, y and the averaged elements $\bar{x}_i$, $\bar{y}$ is

$$x_i = \bar{x}_i + P_{1i}(\bar{x}_m; \bar{y}), \qquad y = \bar{y} + Q_1(\bar{x}_m; \bar{y}) \qquad (28)$$

where $i = 1, 2, \ldots, 5, m = 1, 2, \ldots, 5$. It is noted that the right-hand sides of the transformation are functions of the averaged elements only. The transformations relating the averaged elements $\bar{x}_i$ and the doubly averaged (or secular) elements x_{si} are given by Eqs. (24). It is also noted that the right-hand sides of transformation (24) is a function of the doubly averaged elements only. If the initial conditions for the osculating orbital elements a_o, e_o, i_o, Ω_o, ω_o and M_o are given the initial conditions for the averaged elements $\bar{a}_o$, $\bar{e}_o$, $\bar{i}_o$, $\bar{\omega}_o$, $\bar{\Omega}_o$, and $\bar{M}_o$ can be evaluated from transformation (28) by iteration. The initial conditions for the doubly averaged elements a_{so}, e_{so}, i_{so}, ω_{so}, Ω_{so}, and M_{so} can then be determined from transformation (24).

To illustrate the procedures for computing the orbital elements, first integrate the rate Eqs. (26) for the doubly averaged elements with initial conditions a_{so}, e_{so}, i_{so}, ω_{so}, Ω_{so}, and M_{so}. The first equation of (26) suggests that a_s, e_s, and i_s are constants of motion and take on their initial values. The last three equations of (26) are thus in a form which can be integrated readily. Substituting the integrated values of the doubly averaged orbital elements into transformation (24), the averaged orbital elements would be determined. Likewise, the orbital elements a, e, i, Ω, ω, and M can be evaluated from transformation (28).

V. Comparisons and Comments

This analysis has been developed with an intention to estimate an accurate time history of an artificial satellite perturbed only by the earth gravitational potential through the application of the general theory of the method of averaging. The results obtained in this study have been carefully checked against the solutions given by Kozai, Brouwer, and Arsenault et al.

The comparison with Kozai's theory yields almost exact agreement when his mean elements[6] are introduced into the solution; however, it is noted that the solution for mean anomaly and the explicit transformation relations for computation are absent from his theory. The comparison with Brouwer's theory also indicates excellent agreement except for some second-order differences in the averaged equation for mean anomaly. This difference is believed to be derived from the fact that the Von Zeipel method preserves the canonical form of the averaged equations, while the method of averaging does not. It has been pointed out by Kyner[20] that the Von Zeipel method and the modified canonical method of averaging would, however, lead to the same approximating formulas. The comparison with the theory given by Arsenault suggests the identical difference since the solution for the mean anomaly is adopted from Brouwer's theory.

The solutions given here do not apply to the near-circular, equatorial orbits. For the discussions of the necessary alternative orbital elements, the reader is referred to Ref. 20. Although Eqs. (26) and (27) do not hold for orbits at near critical inclination, Eqs. (18) and (21) are valid.

The extensive application of the general theory of the method of averaging in this perturbation theory is also intended to encourage others to use it for related analyses. Since it requires no assumptions about the conservativeness of the disturbing forces, the method is a rigorous, systematic, and straightforward procedure for studying a dynamical system perturbed by either conservative or nonconservative forces so long as the forcing function satisfies the requirements of the method of averaging. It is also noted, through the discussions of the method of averaging, that this perturbation technique is applicable to a dynamical system with two or more fast variables. Thus, the technique used here can be applied directly in the study of the attitude motion of a spacecraft which has three fast variables (for a first-order theory, see Ref. 21) and other problems in the field of astrodynamics.

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